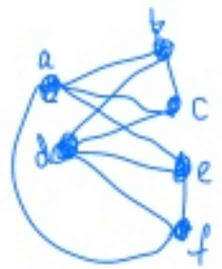
For student making < 90 on the test, you can

① On a separate piece of paper, redo any questions where points are missed.

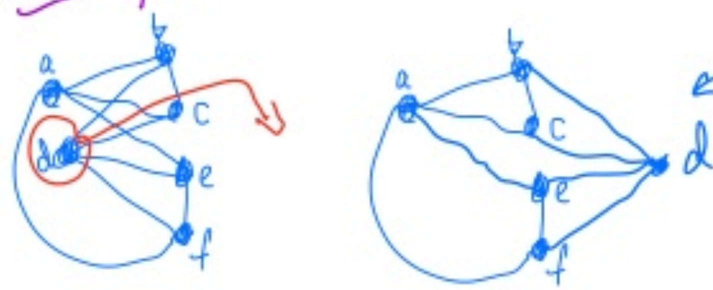
② Explain the math mistakes made.

③ If all this is done correctly, you can get points up to $\frac{1}{2}$ (the distance to 90).

Q: Is this graph planar?



Ans: Yes



← No edges crossing. ✓

Euler characteristic: Given a surface and a graph on the surface, if

v = # of vertices

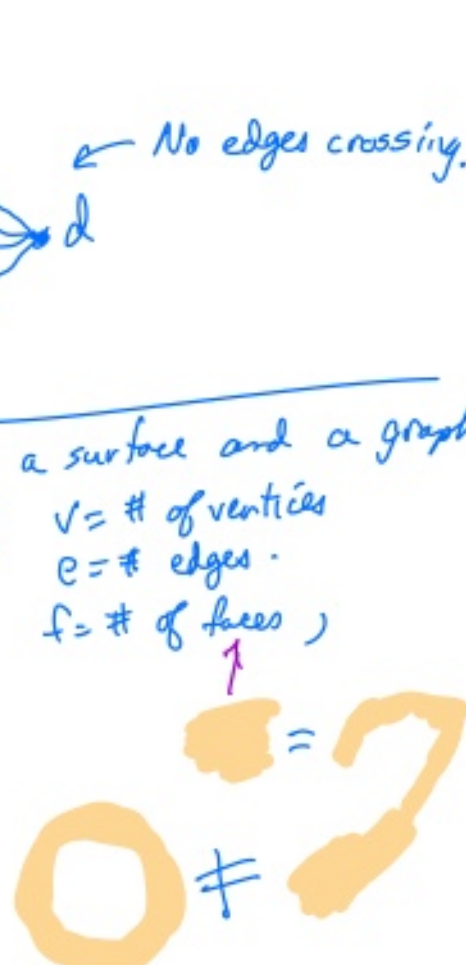
e = # edges.

f = # of faces,

then the number

$$\chi = v - e + f$$

is a number which only depends on the surface and not which graph is used to divide up the surface.



Why is this true?

- Given a graph on a surface, if it is subdivided in some way, χ stays the same.

Why?

on a surface (given).

Ways we can subdivide:

$v,$
$e,$
f

$\binom{\text{new}}{v} = v+1$
$\binom{\text{new}}{e} = e+1$
$\binom{\text{new}}{f} = f$

new $\chi = \chi + 1 - 1 = \chi$

v
e
f

$\binom{\text{new}}{v} = v$
$\binom{\text{new}}{e} = e+1$
$\binom{\text{new}}{f} = f+1$

new $\chi = \chi - 1 + 1 = \chi$.

v
e
f

$\binom{\text{new}}{v} = v$
$\binom{\text{new}}{e} = e+1$
$\binom{\text{new}}{f} = f+1$

new $\chi = \chi - 1 + 1 = \chi$

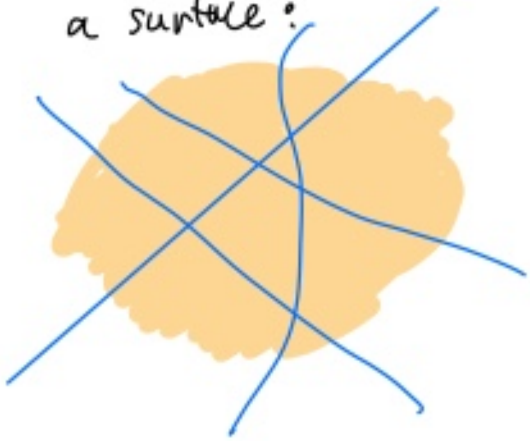
v
e
f

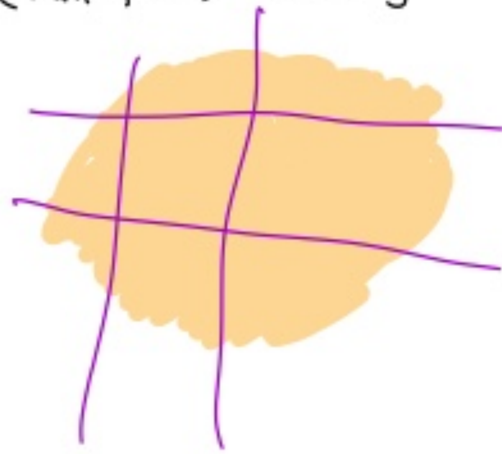
$\binom{\text{new}}{v} = v+1$
$\binom{\text{new}}{e} = e+2$
$\binom{\text{new}}{f} = f+1$

new $\chi = \chi + 1 - 2 + 1 = \chi$.

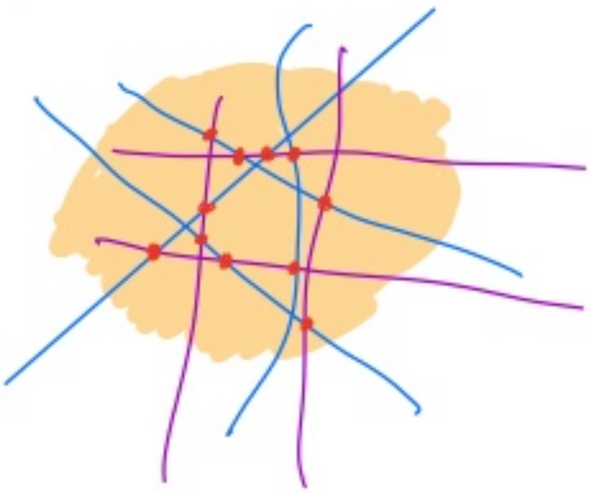
Any subdivision of the graph on the surface can be achieved by a sequence of such operations, and so all subdivisions result in the same χ.

- Now Given any 2 Graphs (with faces) covering a surface:





→ put them on top of each other.



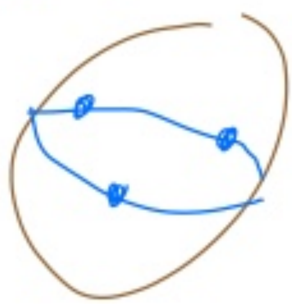
After wiggling, we can make this into one graph by adding vertices

The new graph is a subdivision of both graphs, So the new χ must be the same as both initial graph χ's.

$\therefore$ The Euler characteristic χ is the same for any graph embedded in the surface so that the faces are disks (morphed).

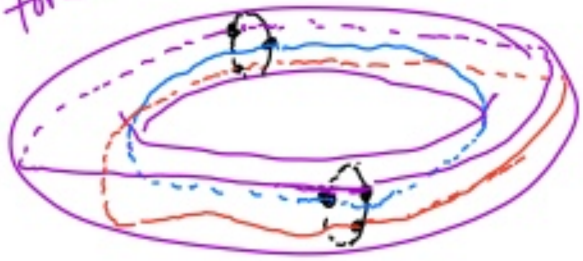
Euler characteristic of the sphere

S^2
sphere



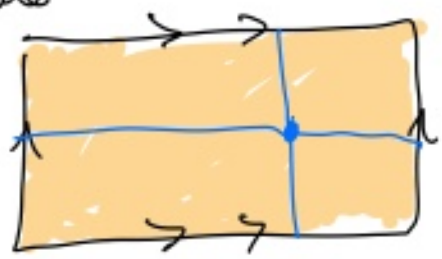
$$\chi = v - e + f$$
$$= 3 - 3 + 2 = \boxed{2}.$$

T^2 torus



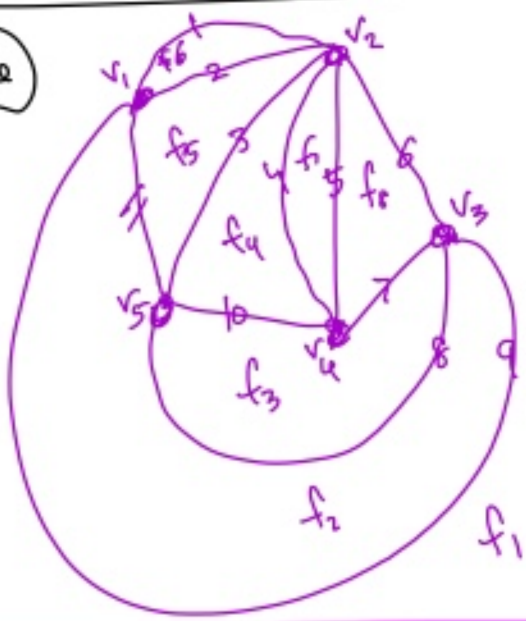
$$\left.\begin{array}{l} v=6 \\ e=12 \\ f=6 \end{array}\right\} \begin{array}{l}\chi = 6-12+6 \\ \quad = 0\end{array}$$

alternate



$$\left.\begin{array}{l} V=1 \\ e=2 \\ f=1 \end{array}\right\} \begin{array}{l}\chi = 1-2+1 \\ \quad = 0.\end{array}$$

Example



$V = 5$

$e = 11$

$f = 8$

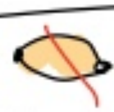
$\chi = v - e + f$

$\quad = 5 - 11 + 8$

$\quad = \boxed{2}$.

<u>Theorem</u>: Suppose we have a planar graph that is simple & connected with v vertices and e edges.

If $v \geq 3$, then $e \leq 3v - 6$.

<u>Proof</u>  Every face must have at least 3 edges on the boundary, because the graph is simple. Every edge is attached to 2 faces.

$\Rightarrow f \leq \frac{e \cdot 2}{3}$

$$\chi = 2 = v - e + f \le v - e + \frac{2e}{3}$$
$$\Rightarrow 2 \le v - \tfrac{1}{3}e$$
$$\Rightarrow 6 \le 3v - e$$
$$\Rightarrow e \le 3v - 6 . \ \square$$

<u>Cor:</u> K_5 is not planar.

K_5 has $v = 5$

$$e = \frac{4 \cdot 5}{2} = 10$$
$$3v - 6 = 15 - 6 = 9 < 10 = e$$

$\therefore$ it does not satisfy the $e \le 3v - 6$.

$\Rightarrow K_5$ is not planar.

<u>Thm</u> (Kuratowski) A connected simple graph is planar $\Leftrightarrow$ it does not contain K_5 or $K_{3,3}$ as a subgraph.
